1. (20 points) Determine whether the following statements are true or false. If true then prove it, and if false then provide a counterexample.

(1) Suppose that the sequences $\{a_n\}$ and $\{b_n\}$ are convergent. Then, $\left\{ \dfrac{a_n}{b_n + 1} \right\}$ is also convergent.

(2) Suppose that a continuous function $f(x)$ is defined on $(1, +\infty)$. Then, $\displaystyle\lim_{x \to +\infty} \frac{1}{x} f(x) = 0$ holds.

Proof for (1). **False.** Let $a_n = -1$ and $b_n = -1 + \frac{1}{n}$. Then, they converges to -1. However, $\frac{a_n}{b_n+1} = -n \to -\infty$. □

Proof for (2). **False.** $f(x) = x$ is continuous, but $\frac{1}{x} f(x) = 1$. □

2. (25 points) Let $a_{n+1} = \left(\frac{a_n}{2}\right)^{\frac{3}{2}}$ and $0 \le a_0 < 8$. Prove that the sequence $\{a_n\}$ is convergent, and the limit is 0.

Proof. We have $a_1 = (a_0/2)^{\frac{3}{2}} \ge 0$. Moreover,

$$(a_1)^2 = (a_0/2)^3 = (a_0)^2(a_0/8) < (a_0)^2,$$

yields $a_1 < a_0 < 8$.

Assume $0 \le a_k < 8$ for some $k \in \mathbb{N}$. Then, we have $a_{k+1} = (a_k/2)^{\frac{3}{2}} \ge 0$. Moreover,

$$(a_{k+1})^2 = (a_k/2)^3 = (a_k)^2(a_k/8) < (a_k)^2,$$

yields $a_{k+1} < a_k < 8$.

Thus, by the mathematical induction, we have $0 \le a_n < a_0 < 8$ for all $n \in \mathbb{N}$ and $a_{n+1} < a_n$. Since a_n is decreasing and bounded below, a_n converges to a limit L by the completeness property. In addition, by the limit location theorem, we have $0 \le L \le a_0 < 8$.

Now, by using $(a_{n+1})^2 = (a_n)^3/8$ and Theorem 5.1, we have

$$L^2 = \lim(a_{n+1})^2 = \lim(a_n)^3/8 = L^3/8.$$

Hence, we have $L^2(L-8) = 0$, namely $L = 0$ or 8. However, $L \le a_0 < 8$ implies $L \ne 8$, and thus $L = 0$.

□

3. (25 points) Let $2a_{n+1} = \dfrac{1}{1+a_n}$ and $a_0 > 0$. Prove that it is convergent, and the limit is $\dfrac{-1+\sqrt{3}}{2}$.

Proof. We first observe $2a_1 = \frac{1}{1+a_0} > 0$. If $a_k > 0$ for some $k \in \mathbb{N}$, then $2a_{k+1} = \frac{1}{1+a_k} > 0$. So, by the mathematical induction, we have $a_n > 0$ for all $n \in \mathbb{N}$. Next,

$$\begin{aligned} |2a_{n+2} - 2a_{n+1}| =& \left| \frac{1}{1+a_{n+1}} - \frac{1}{1+a_n} \right| = \left| \frac{a_{n+1}-a_n}{(1+a_{n+1})(1+a_n)} \right| \\ =& \frac{|a_{n+1}-a_n|}{(1+a_{n+1})(1+a_n)} < |a_{n+1}-a_n|. \end{aligned}$$

Namely, $|a_{n+2} - a_{n+1}| < 2^{-1}|a_{n+1} - a_n|$. Assume

$$|a_{n+k+1} - a_{n+k}| < 2^{-k}|a_{n+1} - a_n|$$

for some $k \in \mathbb{N}$. So, by the mathematical induction, for all $n, m \in \mathbb{N}$

$$|a_{n+m+1} - a_{n+m}| < 2^{-m}|a_{n+1} - a_n|.$$

Therefore, for $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} |a_m - a_n| =& \left| \sum_{k=0}^{m-n-1} (a_{n+k+1} - a_{n+k}) \right| \le \sum_{k=0}^{m-n-1} |a_{n+k+1} - a_{n+k}| \\ \le& \sum_{k=0}^{m-n-1} 2^{-k}|a_{n+1} - a_n| = |a_{n+1} - a_n| \sum_{k=0}^{m-n-1} 2^{-k}. \end{aligned}$$

Since $\sum 2^{-k}$ is a geometric series, we have

$$\sum_{k=0}^{m-n-1} 2^{-k} = \frac{1-2^{-(m-n)}}{1-2^{-1}} < \frac{1}{1-2^{-1}} = 2.$$

Namely,

$$|a_m - a_n| < 2|a_{n+1} - a_n| \le \frac{2|a_1 - a_0|}{2^n}.$$

Define a constant $M = 1 + 2|a_1 - a_0|$. Then, $|a_m - a_n| \le \frac{M}{2^N}$ for $m, n \ge N$.

Given any $\epsilon > 0$, we choose a large natural number $N > \log_2(M/\epsilon)$. Then, we have $|a_m - a_n| < \epsilon$ for $m, n \ge N$. Namely, a_n is a Cauchy sequence, and thus it converges to a limit L.

By using $\frac{1}{2} = a_{n+1}(1 + a_n)$ and Theorem 5.1, we have

$$\frac{1}{2} = \lim a_{n+1}(1+a_n) = \lim a_{n+1} + (\lim a_{n+1})(\lim a_n) = L + L^2.$$

Hence, solving the quadratic equation yields $L = \frac{-1\pm\sqrt{3}}{2}$. Since $a_n > 0$, by the limit location theorem, we have $L \ge 0$. Thus, $L = \frac{-1+\sqrt{3}}{2}$.

□

4. (10 points) Find the radius of convergence of the power series $\sum_{n=0}^{\infty}(n!)x^n$, and explain why.

Proof. Given $x \neq 0$, we define $a_n = (n!)x^n$. Then, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!|x|^{n+1}}{(n!)|x|^n} = (n+1)|x| > 1,$$

for all $n \geq |x|^{-1}$. We choose a natural number N such that $N \geq |x|^{-1}$. Then, for $n > N$ we have

$$|a_n| = |a_N| \prod_{k=N}^{n-1} \left|\frac{a_{k+1}}{a_k}\right| \geq |a_N| \prod_{k=N}^{n-1} 1 = |a_N| > 0,$$

namely a_n can NOT converge to 0. Hence, by the test for divergence, a_n diverges.

On the other hand, if $x = 0$ the power series clearly converges to 0. Thus, the radius of convergence is 0. $\square$

5. (20 points) Continuous functions $f(x)$ and $g(x)$ are defined for $x \in (0, +\infty)$. Let $\{a_n\}_{n\geq 1}$ be a positive increasing sequence tending to $+\infty$ with $a_{n+1} \leq a_n + 1$. Suppose that

$$f(a_n/m) = g(a_n/m)$$

holds for all natural numbers $n, m \in \mathbb{N}$. Then, Prove that $f(x) = g(x)$ holds for all $x \in \mathbb{R}$.

Proof. Since a_n tends to $+\infty$, given $x > 0$ and $m \in \mathbb{N}$ with $m \geq a_1/x$, we have a non-empty set $A_m = \{n \in \mathbb{N} : a_n \leq mx\}$. Since a_n is increasing sequence, A_m is a finite set, and thus we have $n_m = \max A_m \in \mathbb{N}$. Then, by definition of A_m, the following holds

$$a_{n_m} \leq mx < a_{n_m+1}.$$

Now, we set $b_m = a_{n_m}/m$. Then,

$$|x - b_m| = x - \frac{a_{n_m}}{m} \leq \frac{a_{n_m+1}}{m} - \frac{a_{n_m}}{m} \leq \frac{1}{m},$$

namely, $\lim b_m = x$.

On the other hand, $h(x) = f(x) - g(x)$ is a continuous function with $h(b_m) = 0$. Therefore, by Theorem 11.5A, we have

$$h(x) = \lim h(b_m) = \lim 0 = 0,$$

namely, $f(x) = g(x)$. □

6.(10 points, bonus problem) We say that a set $S \subset \mathbb{R}$ is countable if there exists a sequence $\{a_n\}_{n\geq 1}$ such that $S \subset \{a_n : n \in S\}$.

An increasing function $f(x)$ is defined for $x \in (-\infty, +\infty)$. Let S denote the set of points where $f(x)$ is discontinuous. Prove that S is countable.

(You may need to use that facts that given any two different real numbers $x < y$, there exists a rational number r such that $x < r < y$. Moreover, the set $\mathbb{Q}$ of rational numbers is countable.)

Proof. First of all, we claim that the left-hand limit of $f(x)$ exist for each $x \in \mathbb{R}$, and it is less than or equal to $f(x)$.

Given $x_0 \in \mathbb{R}$, $\{f(x_0 - \frac{1}{n})\}_{n\in\mathbb{N}}$ is an increasing sequence with $f(x_0 - \frac{1}{n}) \leq f(x_0)$. Therefore, there exists a number L such that $\lim f(x_0 - \frac{1}{n}) = L$ and $f(x_0 - \frac{1}{n}) \leq L \leq f(x_0)$ by the completeness property and the limit location theorem.

Then, given $\epsilon > 0$, there exists a large natural number N such that $L - f(x_0 - \frac{1}{N}) < \epsilon$. Thus, given $x \in (x_0 - \frac{1}{N}, x_0)$ we have

$$|L - f(x)| \leq |L - f(x_0 - N^{-1})| < \epsilon,$$

namely $\lim\limits_{x\to x_0^-} f(x) = L \leq f(x_0)$.

In the same manner, the right-hand limit of $f(x)$ exist, and it is greater than or equal to $f(x)$. Namely, $\lim\limits_{x\to x_0^-} f(x) \leq \lim\limits_{x\to x_0^+} f(x)$.

Therefore, if $f(x)$ is discontinuous at a point x_1 then

$$\lim_{x\to x_1^-} f(x) < \lim_{x\to x_1^+} f(x),$$

and thus there exists a rational number r_1 such that

$$\lim_{x\to x_1^-} f(x) < r < \lim_{x\to x_1^+} f(x).$$

Moreover, given another point $x_1 < x_2$ where $f(x)$ is discontinuous, we have

$$r_1 < \lim_{x\to x_1^+} f(x) \leq \lim_{x\to x_1^-} f(x) < r_2.$$

Hence, there exists one-to-one correspondence between S and a subset of $A \subset \mathbb{Q}$. Namely, there exists an onto function $g : A \to S$.

On the other hand, the given fact implies that there exists a sequence $\{a_n\}$ such that $A \subset \mathbb{Q} \subset \{a_n : n \in \mathbb{N}\}$. Namely, there exists a subsequence a_{n_j} such that $A = \{a_{n_j} : j \in \mathbb{N}\}$. So, we have an onto function $h : \mathbb{N} \to A$ defined by $h(j) = a_{n_j}$. Then, by using the onto function $g \circ h : \mathbb{N} \to S$, we can define a sequence b_j as $b_j = g(h(j))$. Then, we have $S = \{b_j : n \in \mathbb{N}\}$. Therefore, S is countable. $\square$